Finite-genus solutions for the Hirota's bilinear difference equation.

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Abstract

The finite-genus solutions for the Hirota's bilinear difference equation are constructed using the Fay's identities for the θ-functions of compact Riemann surfaces.

In the present work I want to consider once more the question of constructing the finite-genus solutions for the famous Hirota's bilinear difference equation (HBDE) [1]

$$\tau(k-1,l,m)\,\tau(k+1,l,m)+\tau(k,l-1,m)\,\tau(k,l+1,m)+\tau(k,l,m-1)\,\tau(k,l,m+1)=0 \quad (1)$$

which has been solved in [2] using the so-called algebraic-geometrical approach. This method, which is the most powerful method for deriving the quasi-periodic solutions (QPS) and which has been developed for almost all known integrable systems, exploits some rather sophisticated pieces of the theory of functions of complex variables and is based on some theorems determining the number of functions with prescribed structure of singularities on the Riemann surfaces (see [3] for review). However, in some cases the QPS can be found with less efforts, in a more straightforward way, using the fact that the finite-genus QPS (and namely they are the subject of this note) of all integrable equations possess similar and rather simple structure: up to some phases they are 'meromorphic' combinations of the θ-functions associated with compact Riemann surfaces [4] (the situation resembles the pure soliton case, where solutions are rational functions of exponents). Thus, to construct these solutions we only have to determine some constant parameters. This, as in the pure soliton case, can be done directly, using the well-known properties of the θ-functions of compact Riemann surfaces. In [5] such approach was developed for the Ablowitz-Ladik hierarchy, where the finite-genus solutions were 'extracted' from the so-called Fay's formulae [6, 4]. The fact, that the HBDE is closely related to the Fay's identities is not new and was mentioned, e.g., in [2] (see Remark 2.7), but contrary to this work I will use these identities as a starting point and will show how one can derive from them, by rather short and very simple calculations

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(which to my knowledge have not been presented explicitly in the literature), a wide family of solutions for the HBDE.

Consider a compact Riemann surface X of the genus g. One can choose a set of closed contours (cycles) $\{a_i, b_i\}_{i=1,\dots,g}$ with the intersection indices

$$a_i \circ a_j = b_i \circ b_j = 0, \qquad a_i \circ b_j = \delta_{ij} \qquad i,j = 1,\dots,g \tag{2}$$

and find g independent holomorphic differentials which satisfy the normalization conditions

$$\oint_{a_i} \omega_k = \delta_{ik} \tag{3}$$

The matrix of the b-periods,

$$\Omega_{ik} = \oint_{b_i} \omega_k \tag{4}$$

determines the so-called period lattice, $L_\Omega = \{\boldsymbol{m} + \Omega \boldsymbol{n}, \quad \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^g\}$, the Jacobian of this surface $\mathrm{Jac}(X) = \mathbb{C}^g / L_\Omega$ ($2g$ torus) and the Abel mapping $X \to \mathrm{Jac}(X)$,

$$\boldsymbol{\mathcal{A}}(P) = \int_{P_0}^{P} \boldsymbol{\omega} \tag{5}$$

where $\boldsymbol{\omega}$ is the g-vector of the 1-forms, $\boldsymbol{\omega} = (\omega_1, \dots, \omega_g)^T$ and P_0 is some fixed point of X.

A central object of the theory of the compact Riemann surfaces is the θ-function, $\theta(\boldsymbol{\zeta}) = \theta(\boldsymbol{\zeta}, \Omega)$,

$$\theta(\boldsymbol{\zeta}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} \exp\{\pi i \, (\boldsymbol{n}, \Omega \boldsymbol{n}) \, + 2\pi i \, (\boldsymbol{n}, \boldsymbol{\zeta})\} \tag{6}$$

where $(\boldsymbol{\xi}, \boldsymbol{\eta})$ stands for $\sum_{i=1}^{g} \xi_i \eta_i$, which is a quasiperiodic function on $\mathbb{C}^g$

$$\theta(\boldsymbol{\zeta} + \boldsymbol{n}) = \theta(\boldsymbol{\zeta}) \tag{7}$$

$$\theta(\boldsymbol{\zeta} + \Omega \boldsymbol{n}) = \exp\{-\pi i \, (\boldsymbol{n}, \Omega \boldsymbol{n}) \, - 2\pi i \, (\boldsymbol{n}, \boldsymbol{\zeta})\} \, \theta(\boldsymbol{\zeta}) \tag{8}$$

for any $\boldsymbol{n} \in \mathbb{Z}^g$.

The famous Fay's trisecant formula can be written as

$$\sum_{i=1}^{3} a_i \, \theta(\boldsymbol{\zeta} + \boldsymbol{\eta}_i) \, \theta(\boldsymbol{\zeta} - \boldsymbol{\eta}_i) = 0 \tag{9}$$

where

$$2\boldsymbol{\eta}_1 = -\boldsymbol{\mathcal{A}}(P_1) + \boldsymbol{\mathcal{A}}(P_2) + \boldsymbol{\mathcal{A}}(P_3) - \boldsymbol{\mathcal{A}}(P_4), \tag{10}$$

$$2\boldsymbol{\eta}_2 = \boldsymbol{\mathcal{A}}(P_1) - \boldsymbol{\mathcal{A}}(P_2) + \boldsymbol{\mathcal{A}}(P_3) - \boldsymbol{\mathcal{A}}(P_4), \tag{11}$$

$$2\boldsymbol{\eta}_3 = \boldsymbol{\mathcal{A}}(P_1) + \boldsymbol{\mathcal{A}}(P_2) - \boldsymbol{\mathcal{A}}(P_3) - \boldsymbol{\mathcal{A}}(P_4). \tag{12}$$

Here $P_1, \dots, P_4$ are arbitrary points of X, and the constants a_i are given by

$$
\begin{aligned}
a_1 &= e(P_4, P_1)\, e(P_2, P_3), &(13)\\
a_2 &= e(P_4, P_2)\, e(P_3, P_1), &(14)\\
a_1 &= e(P_4, P_3)\, e(P_1, P_2). &(15)
\end{aligned}
$$

The skew-symmetric function $e(P, Q)$, $e(P, Q) = -e(Q, P)$, is closely related to the prime form [4] and is given by

$$
e(P, Q) = \theta\left[\boldsymbol{\delta}', \boldsymbol{\delta}''\right]\left(\boldsymbol{\mathcal{A}}(Q) - \boldsymbol{\mathcal{A}}(P)\right) \tag{16}
$$

where $\theta\left[\boldsymbol{\alpha}, \boldsymbol{\beta}\right](\boldsymbol{\zeta})$ is the so-called θ-function with characteristics,

$$
\theta\left[\boldsymbol{\alpha}, \boldsymbol{\beta}\right](\boldsymbol{\zeta}) = \exp\left\{\pi i\ (\boldsymbol{\alpha}, \Omega\boldsymbol{\alpha}) + 2\pi i\ (\boldsymbol{\alpha}, \boldsymbol{\zeta} + \boldsymbol{\beta})\right\} \theta\left(\boldsymbol{\zeta} + \Omega\boldsymbol{\alpha} + \boldsymbol{\beta}\right), \tag{17}
$$

and $(\boldsymbol{\delta}', \boldsymbol{\delta}'') \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is a non-singular odd characteristics,

$$
\theta\left[\boldsymbol{\delta}', \boldsymbol{\delta}''\right](\mathbf{0}) = 0, \qquad \mathrm{grad}_{\boldsymbol{\zeta}}\, \theta\left[\boldsymbol{\delta}', \boldsymbol{\delta}''\right](\mathbf{0}) \neq \mathbf{0} \tag{18}
$$

Now it is very easy to establish relations between (9) and the HBDE (1). To do this one has first to introduce the discrete variables by

$$
\Theta(k, l, m) = \theta\left(\boldsymbol{\zeta} + k\boldsymbol{\eta}_1 + l\boldsymbol{\eta}_2 + m\boldsymbol{\eta}_3\right) \tag{19}
$$

The Fay's identity (9) can now be rewritten as

$$
\begin{aligned}
a_1\, \Theta(k-1, l, m)\, \Theta(k+1, l, m) + a_2\, \Theta(k, l-1, m)\, \Theta(k, l+1, m) & \\
+ a_3\, \Theta(k, l, m-1)\, \Theta(k, l, m+1) &= 0
\end{aligned} \tag{20}
$$

from which it follows that the quantity

$$
\tau(k, l, m) = a_1^{k^2/2}\, a_2^{l^2/2}\, a_3^{m^2/2}\, \Theta(k, l, m) \tag{21}
$$

satisfies HBDE (1).

Thus the last formula,

$$
\tau(k, l, m) = a_1^{k^2/2}\, a_2^{l^2/2}\, a_3^{m^2/2}\, \theta\left(\boldsymbol{\zeta} + k\boldsymbol{\eta}_1 + l\boldsymbol{\eta}_2 + m\boldsymbol{\eta}_3\right) \tag{22}
$$

where θ is the θ-function of some compact Riemann surface X and the constants a_i and η_i depend on four points (paths) on this surface, determines a family of finite-genus solutions of the HBDE.

Now I want to discuss the solutions obtained above. First of all it should be noted that these solutions are 'finite-genus' but not quasiperiodic. At first glance, this seems to be strange, because usually the finite-genus solutions naturally appear when one solves quasiperiodic problems. For, example, in $1+1$ dimensional discrete systems, such as Toda chain, Ablowitz-Ladik equations, etc, the quasiperiodicity leads to the polynomial dependence of the scattering matrix of the auxiliary problem on the spectral parameter. These polynomials determine some hyperelliptic curve (spectral curve) of finite genus, and the quasiperiodic solutions are built up of the θ-functions corresponding to the latter. Thus, in some sense, in discrete systems the quasiperiodicity implies the 'finite-genus' property. In

other words, quasiperiodic solutions are finite-genus. However, the reverse is not obligatory true. The Fay's identities do not imply that the integrals $\boldsymbol{\mathcal{A}}(P)$ are in some way related to the periods $\oint \boldsymbol{\omega}$. Formula (9) determine, so to say, 'local' properties of the θ-functions, and all above consideration was local, without reference to some boundary conditions (quasiperiodicity).

Another point which I would like to discuss here is to compare the approach of this note with the algebro-geometrical one. The ideology of the latter is to operate on the Riemann surface: the key moment in calculating the Baker-Akhiezer function, the central object of the algebro-geometrical method, is to satisfy the condition that it is a single-valued function of the point of the Riemann surface. Since we didn't introduce the Baker-Akhiezer function and didn't study its analytical properties the question of how our solutions depend on the points $P_i \in X$ (or on the integral paths from P_0 to P_i, to be more precise) is not so crucial as in the algebro-geometrical approach and is in some sense the question of parametrization of constants. One can restrict the points P_i, $i = 1, 2, 3$, to some neighborhood of the point P_0 and rewrite the Abel's integrals in terms of local coordinates (with some polynomial representing the Riemann surface). This enables not to mention the Riemann surface and reformulate all results in terms of integrals over the complex plane. As to the 'global' (or 'homotopical') effects, which arise when we add to the paths (P_0, P_i) some integer cycles $(\sum_{k=1}^{g} m_k a_k + n_k b_k,\ m_k, n_k \in \mathbb{Z})$ it should be noted that, if we consider the Abel integral as mapping $X \to \mathbb{C}^g$, then such deformations of the contours change $\boldsymbol{\mathcal{A}}(P)$ as $\boldsymbol{\mathcal{A}}(P) \to \boldsymbol{\mathcal{A}}(P) + \boldsymbol{\gamma}$, $\boldsymbol{\gamma} \in L_\Omega$. This results first in adding some *half-period* to the argument of the θ-function in (22),

$$\boldsymbol{\zeta} + \sum_{i=1}^{3} k_i \boldsymbol{\eta}_i \to \boldsymbol{\zeta} + \sum_{i=1}^{3} k_i \boldsymbol{\eta}_i + \frac{1}{2} \sum_{i=1}^{3} k_i \boldsymbol{\gamma_i}, \qquad \boldsymbol{\gamma_i} \in L_\Omega \tag{23}$$

and, second, in altering the constants a_i's, $a_i \to a_i \exp(f_i)$. Thus, we come to the point where one can try to apply the theory of Backlund transformations for the HBDE to describe the transformations of θ-functions due to half-period shifts (and vice versa). This is an interesting problem, which deserves special studies.

To conclude I would like to note the following. In principle, the direct approach based on the Fay's identities can be used to derive the finite-genus solutions not only for the HBDE but for almost all known integrable systems (some examples one can find in the book [4]). However, contrary to the case of the HBDE where all 'calculations' take only two lines (formulae (19) and (21) above), in the case of partial differential equations such as, e.g., KP equation the corresponding calculations become rather cumbersome. For example, to solve the KP equation one has to expand the Fay's identities up to the third order in some small parameter. From the other hand, it is a widely known fact that almost all known integrable equations can be derived from the HBDE. Hence one can 'skip' the Fay's identities and use solutions (22) as a starting point. In [5] it was shown how to obtain some finite-genus solutions for the nonlinear Schrödinger and KP equations using the corresponding solutions for the Ablowitz-Ladik hierarchy, which can be viewed as some 'pre-continuous' version of the HBDE (in the Ablowitz-Ladik hierarchy 2 of 3 discrete coordinates of the HBDE are presented as the Miwa's shifts of two infinite sets of continuous variables).

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